air/fluid mass elements. This is not a correct interpretation because (x, y, z) are the coordinates of the mass element and their time derivatives should be the velocity of the fluid mass at inlet and the outlet.

To conclude, there is no need to treat the air-breathing engine separately by including $\dot{m}$ terms. Its effect is completely accounted for by using the thrust attributable to the engine [which is equal to $-\dot{m}(\dot{\bar{R}}+\dot{\bar{\rho}})$] for computing components of external force and torque to be used on the right-hand side of equations of motion given in Tables 1 and 2 of Ref. 1.

Reference

[1]Bilimoria, K. D., and Schmidt, D. K., "Integrated Development of the Equations of Motion for Elastic Hypersonic Flight Vehicles," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 1, 1995, pp. 73–81.

Reply by the Authors to Nivritti Vithoba Kadam

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Introduction

In our paper,[1] a Lagrangian approach is used to determine the dynamics of rigid-body motion, elastic deformation, fluid flow, rotating machinery, wind, and a spherical rotating Earth model and to account for their mutual interactions. The preceding Technical Comment claims that the terms containing mass flow rates in the equations of motion of Ref. 1 are erroneous and presents an analysis using a Newtonian approach to support this claim. We stand by the equations of motion presented in Ref. 1 and note that the claim and many other statements made in the Technical Comment arise primarily from a misinterpretation of certain velocity terms.

Contents

Using a Newtonian approach, Meriam[2] gives a derivation of force and moment equations for variable-mass systems. For the case where the mass of the system is distributed over a finite volume and the system may have any general motion, the force equation is given by Eq. (173) of Ref. 2, reproduced as follows:

$$\sum \bar{F} = m\ddot{\bar{r}}_e - \dot{m}\bar{u} - \frac{d^2}{dt^2}(m\bar{\rho}_e) \tag{1}$$

Equation (1) is written in the nomenclature of Ref. 2; using the nomenclature of Ref. 1, this equation is written as

$$m\bar{g} + \bar{F}_A + A_{\text{open}}(P - P_\infty)\hat{n}_{\text{open}} = m\ddot{\bar{E}}_{\text{open}} - \dot{m}\bar{V}_{f,0} - \frac{d^2}{dt^2}(m\bar{r}_{\text{open}}) \tag{2}$$

Noting that $\dot{m} = \dot{m}_{\text{fluid}}$, $\bar{E}_{\text{open}} = \bar{R} + \bar{r}_{\text{open}}$, and expanding the last term on the right-hand side of Eq. (2) results in

$$\begin{aligned} &m\bar{g} + \bar{F}_A + A_{\text{open}}(P - P_\infty)\hat{n}_{\text{open}} \\ &\quad = m\ddot{\bar{R}} - \dot{m}_{\text{fluid}}\bar{V}_{f,0} - 2\dot{m}_{\text{fluid}}\dot{\bar{r}}_{\text{open}} - \ddot{m}_{\text{fluid}}\bar{r}_{\text{open}} \end{aligned} \tag{3}$$

Noting that the time derivatives of $\bar{R}$ and $\bar{r}_{\text{open}}$ are evaluated in the inertial frame, that $\bar{V}_I = (d\bar{R}/dt)|_I$, and that fluid flow takes place through several openings in the vehicle, one obtains

$$\begin{aligned} &m\bar{g} + \bar{F}_A + \sum_{\text{openings}} [A_{\text{open}}(P - P_\infty)\hat{n}_{\text{open}} + \dot{m}_{\text{fluid}}\bar{V}_{f,0}] \\ &\quad = m\frac{d\bar{V}_I}{dt}\bigg|_I - \sum_{\text{openings}} \left[2\dot{m}_{\text{fluid}}\frac{d\bar{r}_{\text{open}}}{dt}\bigg|_I + \ddot{m}_{\text{fluid}}\bar{r}_{\text{open}}\right] \end{aligned} \tag{4}$$

Using the definition of $\bar{F}_T$ given by Eq. (35) of Ref. 1 and applying Eq. (1) of Ref. 1 to $\bar{r}_{\text{open}}$, Eq. (4) can be written as

$$\begin{aligned} m\frac{d\bar{V}_I}{dt}\bigg|_I &= m\bar{g} + \bar{F}_A + \bar{F}_T \\ &\quad + \sum_{\text{openings}} \left[2\dot{m}_{\text{fluid}}\left(\frac{d\bar{r}_{\text{open}}}{dt}\bigg|_B + \bar{\omega}_{B,I} \times \bar{r}_{\text{open}}\right) + \ddot{m}_{\text{fluid}}\bar{r}_{\text{open}}\right] \end{aligned} \tag{5}$$

Equation (5) is the same as Eq. (41) of Ref. 1, which was independently developed using a Lagrangian approach. The left-hand side of this equation can be expanded as described in Ref. 1; the resulting vector force equation can be written as the three scalar force equations presented in Table 1 of Ref. 1.

Similarly, starting from the vector moment equation given by Eq. (174) of Ref. 2, we obtain Eq. (46) of Ref. 1, which in turn leads to the three scalar moment equations presented in Table 2 of Ref. 1.

Reference 1 defines the quantity $\bar{r}_{\text{open}}$ as the average location of an opening (inlet or outlet) in the vehicle, relative to the origin of the body frame (vehicle center of mass). The time derivative of $\bar{r}_{\text{open}}$ evaluated in the body frame, $(d\bar{r}_{\text{open}}/dt)|_B$, is the velocity of the opening relative to the body frame; this term arises from elastic motion of the vehicle, and from changes in vehicle c.m. location because of fuel consumption. The body axes scalar components of the vector $(d\bar{r}_{\text{open}}/dt)|_B$ for inlets and outlets are $(\dot{x}_{\text{in}}, \dot{y}_{\text{in}}, \dot{z}_{\text{in}})$ and $(\dot{x}_{\text{out}}, \dot{y}_{\text{out}}, \dot{z}_{\text{out}})$, respectively. It is emphasized that these quantities represent velocities because of the elastic motion of the vehicle mass elements at openings and changes in vehicle c.m. location because of fuel consumption; they are not velocities of fluid mass elements at openings.

As defined in Ref. 1, the velocity of fluid mass elements relative to an opening is $\bar{V}_{f,0}$. It is noted that this velocity only appears implicitly in the force and moment equations presented in Ref. 1 through the thrust terms $\bar{F}_T$ and $\bar{M}_T$ defined in Eqs. (35) and (38), respectively, of Ref. 1.

There is no anomaly indicated by the observation that terms containing $\ddot{m}_{\text{fluid}}$ will disappear if the origin is taken at an opening. Reference 1 defines the origin of the body frame at the instantaneous mass center of the vehicle. Also, Ref. 2 states that "the linear momentum of a time-varying mass depends on the location with respect to the mass center of the position where mass is added (or subtracted)."

In the development of the point-mass model, the explicitly appearing fluid flow terms in the force equation [see Eq. (5)] have been dropped. The primary contribution of these terms results from the vehicle angular velocity $\bar{\omega}_{B,I}$, which is not defined in the point-mass model; the secondary contributions resulting from elastic effects, changes in vehicle c.m. location because of fuel consumption, and unsteady mass flow effects have been neglected. The mass flow rate term implicit in the thrust force $\bar{F}_T$ has, of course, been retained.

Finally, the corrected force and moment equations derived in the Technical Comment [Eqs. (10) and (11)] are themselves erroneous. For a variable mass system, the resultant of external forces/moments is not just equal to the time rate of change of linear/angular momentum as stated [see Eq. (2) and an unnumbered equation preceding Eq. (3)] in the Technical Comment. The resultant of external forces/moments equals the time rate of change of the linear/angular momentum of the varying mass minus the rate at which linear/angular momentum is being changed by the mass elements entering and leaving the system; see Eqs. (171) and (174) of Ref. 2. This principle is also used in the derivation of the Navier–Stokes equations,[3] which are based on conservation of momentum and are consistent with Newton's laws.

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In conclusion, we stand by our development of the equations of motion, and therefore no changes or corrections to Ref. 1 are warranted at this time.

References

[1]Bilimoria, K. D., and Schmidt, D. K., "Integrated Development of the Equations of Motion for Elastic Hypersonic Flight Vehicles," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 1, 1995, pp. 73–81.

[2]Meriam, J. L., *Statics and Dynamics*, Wiley, New York, 1966, pp. 315–321.

[3]Bertin, J. J., and Smith, M. L., *Aerodynamics for Engineers*, Prentice–Hall, Englewood Cliffs, NJ, 1989.

Errata

Feedback Design for Unstable Plants with Saturating Nonlinearities: Single-Input, Single-Output

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EQUATIONS (17–21) should be corrected as follows:

Equation (17) should read

$$d_n(t) = y(t) - y_L(t)$$

The first and third of Eqs. (18) should read

$$E(s) = -\frac{D_n(s)[1 + L_n(s)]L(s)}{s^n G(s)[1 + L(s)]} + \frac{R(s)}{1 + L(s)}$$

$$e_{ss}(t) = \lim_{s \to 0}\left[-\frac{D_n(s)[1 + L_n(s)]L(s)}{s^{n-1} G(s)[1 + L(s)]} + \frac{sR(s)}{1 + L(s)}\right]$$

The first of Eqs. (19) should read

$$q^{sE}|_{1^{\mathrm{st}}\ \mathrm{term}} = [q^{D_n} + n - 1 + q^P]$$

The first of Eqs. (20) should read

$$-q^{D_n} > n - 1 + q^P$$

The statement following Eqs. (20) should read as follows:

If $q^{L_n} > n - 1 + q^P$, then a necessary and sufficient condition for the signal D_n in the stable system of Fig. 4 to produce zero steady-state error is $q^{D_n} \le -q^{L_n}$. Taking the conservative $q^{D_n} = -q^{L_n}$, Eqs. (20) become

$$q^{L_n} > n - 1 + q^P \tag{21a}$$

$$q^L > q^R - 1 \tag{21b}$$